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Description of surfaces associated with $\mathbb{C}P^{N-1}$ sigma models on Minkowski space

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Abstract

The objective of this paper is to construct and investigate smooth orientable surfaces in \mathbb{R}^{N^2-1} by analytical methods. The structural equations of surfaces in connection with $\mathbb{C}P^{N-1}$ sigma models on Minkowski space are studied in detail. This is carried out using moving frames adapted to surfaces immersed in the $su(N)$ algebra. The first and second fundamental forms of these surfaces as well as the relations between them as expressed in the Gauss–Weingarten and Gauss–Codazzi–Ricci equations are found. The Gaussian curvature, the mean curvature vector and the Willmore functional expressed in terms of a solution of $\mathbb{C}P^{N-1}$ sigma model are obtained. An example of a surface associated with the $\mathbb{C}P^1$ model is included as an illustration of the theoretical results.

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1. Introduction

Over the last few decades surfaces immersed in multi-dimensional spaces have been extensively researched in connection with harmonic maps and integrable systems (for a review, see e.g. [1] and the references therein). To give the proper credit, we shall mention that some of the results were in fact rediscovered—they can be found in the 19th century works by Enneper and his collaborators [2,3] and Dobriner [4] who were the first to find some solutions of structural equations describing minimal surfaces. Similarly, the Willmore functional, a key notion in modern global analysis of surfaces, was in fact already used in 1923 by Thomsen and Shadow [5], who also pioneered the use of harmonic maps (so named only in the 1950s) as a tool for construction of conformally parameterized surfaces in \mathbb{R}^3 . Accounts of the complicated history of these discoveries and rediscoveries can be found, e.g. in [6,7].

The motivation for the contemporary research in this area came largely from applications in various branches of physical, biological and chemical sciences as well as from engineering. The progress in the analytical treatment of surfaces obtained from nonlinear differential equations has been rapid and resulted in many new techniques and theoretical approaches. Some of the most interesting developments have been in the study of surfaces immersed in Lie algebras, using techniques of completely integrable systems, e.g. Bäcklund and Darboux transformations, theory of solitons [1,6,8–13]. These surfaces are characterized by fundamental forms whose coefficients satisfy the Gauss–Weingarten and the Gauss–Codazzi–Ricci equations.

In this work we apply a group-theoretical approach to surfaces associated with $\mathbb{C}P^{N-1}$ sigma models. This line of investigation was initiated in [14,15] where it was shown that two-dimensional constant mean curvature surfaces in three- and eight-dimensional spaces are associated with the $\mathbb{C}P^1$ and $\mathbb{C}P^2$ sigma models defined on Euclidean spaces. Further, it was demonstrated in [16] that any surface described by $\mathbb{C}P^{N-1}$ models on Euclidean space can be constructed by a choice of a moving frame based on $su(N)$ algebra representation parameterized by a corresponding solution of the model. This has been a significant result since surfaces immersed in Lie algebras are known to show up in many physical systems (see, e.g. [17,18]). Our objective in this paper is to extend this approach to the case of $\mathbb{C}P^{N-1}$ sigma models defined on Minkowski space. To this end we have devised a new technique for construction of a moving frame; the properties of surfaces obtained in this way, e.g. curvatures, turned out to be significantly different from the ones in the case of sigma models on Euclidean space.

The use of sigma models in mathematical physics has encompassed predominantly models defined on Euclidean spaces, since a great number of physical systems can be reduced to these models. However, in recent literature we find an increasing number of examples when reductions lead to sigma models on Minkowski space and the need for description of surfaces related to these models are certainly there. One such example is the string theory in which sigma models on space-time and their supersymmetric extensions play a crucial role (e.g. [19,20]). Classical configuration of strings can be described by common solutions of the Nambu–Goto–Polyakov action and a system of Dirac type equations intimately connected to $\mathbb{C}P^{N-1}$ models [21,22]. Other relevant applications of recent interest are in the areas of statistical physics (e.g. reduction of self-dual Yang–Mills equations to the Ernst

model [23,24]), phase transitions (e.g. dynamics of vortex sheets, growth of crystals, surface waves, etc., [17,18]) and the theory of fluid membranes [25–27]. In this last example it is known that the free energy per molecule is determined by two invariants (the mean and Gaussian curvatures) of a surface associated with particular class of solutions of $\mathbb{C}P^{N-1}$ sigma model defined on Minkowski space.

2. $\mathbb{C}P^{N-1}$ sigma models and their Euler–Lagrange equations

As a starting point let us present some basic formulae and notation for $\mathbb{C}P^{N-1}$ sigma models defined on Minkowski space.

The points of the complex coordinate space \mathbb{C}^N will be denoted by $z = (z_1, \dots, z_N)$ and the Hermitian inner product in \mathbb{C}^N by

$$\langle z, w \rangle = z^\dagger w = \sum_{j=1}^N \bar{z}_j w_j. \tag{2.1}$$

The complex projective space $\mathbb{C}P^{N-1}$ is defined as a set of one-dimensional subspaces in \mathbb{C}^N . The manifold structure on it is defined by an open covering

$$U_k = \{[z] \mid z \in \mathbb{C}^N, z_k \neq 0\}, \quad k = 1, \dots, N,$$

where $[z] = \text{span}\{z\}$, and coordinate maps

$$\varphi_k : U_k \rightarrow \mathbb{C}^{N-1}, \quad \varphi_k(z) = \left(\frac{z_1}{z_k}, \dots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \dots, \frac{z_N}{z_k} \right).$$

Let ξ^1, ξ^2 be the standard Minkowski coordinates in \mathbb{R}^2 , with the metric

$$ds^2 = (d\xi^1)^2 - (d\xi^2)^2.$$

In what follows we suppose that $\xi_L = \xi^1 + \xi^2$, $\xi_R = \xi^1 - \xi^2$ are the light-cone coordinates in \mathbb{R}^2 , i.e.

$$ds^2 = d\xi_L d\xi_R. \tag{2.2}$$

We shall denote by ∂_L and ∂_R the derivatives with respect to ξ_L and ξ_R , respectively, i.e.

$$\partial_L = \frac{1}{2}(\partial_{\xi^1} + \partial_{\xi^2}), \quad \partial_R = \frac{1}{2}(\partial_{\xi^1} - \partial_{\xi^2}).$$

Let us assume that Ω is an open, connected and simply connected subset in \mathbb{R}^2 with Minkowski metric (2.2). In the study of $\mathbb{C}P^{N-1}$ sigma models we are interested in maps

$[z] : \Omega \rightarrow \mathbb{C}P^{N-1}$ which are stationary points of the action functional (see, e.g. [28])

$$S = \frac{1}{4} \int_{\Omega} (D_{\mu}z)^{\dagger} (D^{\mu}z) d\xi^1 d\xi^2, \quad z^{\dagger} \cdot z = 1. \tag{2.3}$$

The covariant derivatives D_{μ} act on $z : \Omega \rightarrow \mathbb{C}^N$ according to the formula

$$D_{\mu}z = \partial_{\mu}z - (z^{\dagger} \cdot \partial_{\mu}z)z, \quad \partial_{\mu} \equiv \partial_{\xi^{\mu}}, \mu = 1, 2 \tag{2.4}$$

and ensure that the action depends only on $[z] : \Omega \rightarrow \mathbb{C}P^{N-1}$ and not on the choice of a representative of the class $[z]$. Thus, the map $[z]$ is determined as a solution of the Euler–Lagrange equations defined by the action (2.3). Writing

$$z = \frac{f}{|f|}, \quad |f| = \sqrt{f^{\dagger}f} \tag{2.5}$$

one can present the action functional (2.3) also in the form

$$S = \int_{\Omega} \frac{1}{4|f|^2} (\partial_L f^{\dagger} P \partial_R f + \partial_R f^{\dagger} P \partial_L f) d\xi_L d\xi_R, \tag{2.6}$$

where the $N \times N$ matrix

$$P = \mathbf{1} - \frac{1}{|f|^2} f \otimes f^{\dagger} \tag{2.7}$$

is an orthogonal projector on \mathbb{C}^N , i.e. $P^2 = P, P^{\dagger} = P$.

It is useful to recall that the action (2.6) has the local (gauge) $U(1) \times \mathbb{R}$ symmetry

$$f \rightarrow e^{i\alpha(\xi_L, \xi_R) + \beta(\xi_L, \xi_R)} f, \quad \alpha(\xi_L, \xi_R), \beta(\xi_L, \xi_R) : \Omega \rightarrow \mathbb{R} \tag{2.8}$$

corresponding to the fact that the model is defined on $\mathbb{C}P^{N-1}$. In addition, the action (2.6) has the $U(N)$ global symmetry

$$f \rightarrow \Phi f, \quad \Phi \in U(N). \tag{2.9}$$

It is also invariant under the conformal transformations

$$\xi_L \rightarrow \alpha(\xi_L), \quad \xi_R \rightarrow \beta(\xi_R), \tag{2.10}$$

where $\alpha, \beta : \mathbb{R} \rightarrow \mathbb{R}$ are arbitrary 1-to-1 maps such that $\partial_L \alpha(\xi_L) \neq 0, \partial_R \beta(\xi_R) \neq 0$, as well as under the parity transformation

$$\xi_L \rightarrow \xi_R, \quad \xi_R \rightarrow \xi_L. \tag{2.11}$$

Let us note that the invariance properties (2.8)–(2.11) are naturally reproduced on the level of Euler–Lagrange equations.

Computing the extremals of the action (2.6), one finds the Euler–Lagrange equations in terms of f

$$P \left\{ \partial_L \partial_R f - \frac{1}{f^{\dagger}f} ((f^{\dagger} \partial_R f) \partial_L f + (f^{\dagger} \partial_L f) \partial_R f) \right\} = 0. \tag{2.12}$$

They can be also expressed in the matrix form

$$[\partial_L \partial_R P, P] = 0 \tag{2.13}$$

or in the form of a conservation law

$$\partial_L[\partial_R P, P] + \partial_R[\partial_L P, P] = 0. \tag{2.14}$$

By explicit calculation one can check that the real-valued functions

$$J_L = \frac{1}{f^\dagger f} \partial_L f^\dagger P \partial_L f, \quad J_R = \frac{1}{f^\dagger f} \partial_R f^\dagger P \partial_R f \tag{2.15}$$

satisfy

$$\partial_L J_R = \partial_R J_L = 0 \tag{2.16}$$

for any solution f of the Euler–Lagrange equations (2.12). The functions J_L and J_R are invariant under local $U(1) \times \mathbb{R}$ and global $U(N)$ transformations (2.8) and (2.9).

3. Surfaces obtained from $\mathbb{C}P^{N-1}$ sigma model

Let us now discuss the analytical description of a two-dimensional smooth orientable surface \mathcal{F} immersed in the $su(N)$ algebra, associated with $\mathbb{C}P^{N-1}$ sigma model (2.12). We shall construct an exact $su(N)$ -valued 1-form whose “potential” 0-form defines the surface \mathcal{F} . Next, we shall investigate the geometric characteristics of the surface \mathcal{F} .

Let us introduce a scalar product

$$(A, B) = -\frac{1}{2} \text{tr} AB$$

on $su(N)$ and identify the $(N^2 - 1)$ -dimensional Euclidean space with the $su(N)$ algebra

$$\mathbb{R}^{N^2-1} \simeq su(N).$$

We denote

$$M_L = [\partial_L P, P], \quad M_R = [\partial_R P, P], \tag{3.1}$$

or, equivalently, using (2.7)

$$M_D = \frac{1}{f^\dagger f} (P \partial_D f \otimes f^\dagger - f \otimes \partial_D f^\dagger P) \in su(N), \quad D = L, R. \tag{3.2}$$

It follows from (2.14) that if f is a solution of the Euler–Lagrange equations (2.12) then

$$\partial_L M_R + \partial_R M_L = 0. \tag{3.3}$$

Therefore, we can identify tangent vectors to the surface \mathcal{F} with the matrices M_L and M_R , as follows:

$$\mathcal{X}_L = M_L, \quad \mathcal{X}_R = -M_R. \tag{3.4}$$

Eq. (3.3) implies there exists a closed $su(N)$ -valued 1-form on Ω

$$\mathcal{X} = \mathcal{X}_L d\xi_L + \mathcal{X}_R d\xi_R, \quad d\mathcal{X} = 0.$$

Because \mathcal{X} is closed and Ω is connected and simply connected, \mathcal{X} is also exact. In other words, there exists a well-defined $su(N)$ -valued function X on Ω such that $\mathcal{X} = dX$. The matrix function X is unique up to addition of any constant element of $su(N)$ and we identify the elements of X with the coordinates of the sought-after surface \mathcal{F} in \mathbb{R}^{N^2-1} . Consequently, we get

$$\partial_L X = \mathcal{X}_L, \quad \partial_R X = \mathcal{X}_R. \tag{3.5}$$

The map X is called the Weierstrass formula for immersion (an analog of formula firstly introduced in the case of minimal surfaces in [29,2]). In practice, the surface \mathcal{F} is found by integration

$$\mathcal{F} : X(\xi_L, \xi_R) = \int_{\gamma(\xi_L, \xi_R)} \mathcal{X} \tag{3.6}$$

along any curve $\gamma(\xi_L, \xi_R)$ in Ω connecting the point $(\xi_L, \xi_R) \in \Omega$ with an arbitrary chosen point $(\xi_L^0, \xi_R^0) \in \Omega$.

By computation of traces of $\mathcal{X}_B \cdot \mathcal{X}_D$, $B, D = L, R$, we immediately find the components of the induced metric on the surface \mathcal{F}

$$G = \begin{pmatrix} G_{LL}, G_{LR} \\ G_{LR}, G_{RR} \end{pmatrix} = \begin{pmatrix} J_L & -\Re \left(\frac{\partial_R f^\dagger P \partial_L f}{f^\dagger f} \right) \\ -\Re \left(\frac{\partial_R f^\dagger P \partial_L f}{f^\dagger f} \right) & J_R \end{pmatrix}. \tag{3.7}$$

The first fundamental form of the surface \mathcal{F} is

$$I = J_L d\xi_L^2 - 2\Re \left(\frac{\partial_R f^\dagger P \partial_L f}{f^\dagger f} \right) d\xi_L d\xi_R + J_R d\xi_R^2. \tag{3.8}$$

In order to establish conditions on a solution f of the Euler–Lagrange equations (2.12) under which the surface exists, we employ the Schwarz inequality

$$|\langle a, Ab \rangle|^2 \leq \langle a, Aa \rangle \langle b, Ab \rangle \tag{3.9}$$

valid for any positive Hermitian operator A (see, e.g. [30]). Also note that equality in (3.9) holds only if there exists $\alpha \in \mathbb{C}$ such that either $\langle \alpha a + b, A(\alpha a + b) \rangle = 0$ or $\langle a + \alpha b, A(a + \alpha b) \rangle = 0$ holds. We may write

$$J_D = \frac{\langle \partial_D f, P \partial_D f \rangle}{\langle f, f \rangle} \geq 0, \quad D = L, R \tag{3.10}$$

and

$$\det G = \frac{\langle \partial_L f, P \partial_L f \rangle \langle \partial_R f, P \partial_R f \rangle - (\Re \langle \partial_L f, P \partial_R f \rangle)^2}{\langle f, f \rangle^2} \geq 0 \tag{3.11}$$

since

$$\langle \partial_L f, P\partial_L f \rangle \langle \partial_R f, P\partial_R f \rangle \geq |\langle \partial_L f, P\partial_R f \rangle|^2 \geq (\Re \langle \partial_L f, P\partial_R f \rangle)^2.$$

Therefore, the first fundamental form I defined by (3.8) is positive for any solution f of the Euler–Lagrange equations (2.12).

Analyzing the cases when equalities in (3.10) and (3.11) hold, we find that I is positive definite in the point (ξ_L^0, ξ_R^0) either if the inequality

$$\Im(\partial_L f^\dagger(\xi_L^0, \xi_R^0) P\partial_R f(\xi_L^0, \xi_R^0)) \neq 0 \tag{3.12}$$

holds or if the vectors

$$\partial_L f(\xi_L^0, \xi_R^0), \partial_R f(\xi_L^0, \xi_R^0), f(\xi_L^0, \xi_R^0) \tag{3.13}$$

are linearly independent. Therefore, any of the conditions (3.12) and (3.13) is a sufficient condition for the existence of the surface \mathcal{F} associated with the solution f of the Euler–Lagrange equations (2.12) in the vicinity of the point (ξ_L^0, ξ_R^0) . If neither of the conditions (3.12) and (3.13) is met on an image $\text{Im}_X(\Theta)$ of a lower-dimensional subset $\Theta \subset \Omega$ then the surface \mathcal{F} may or may not exist, depending on circumstances. If both conditions (3.12) and (3.13) are violated in the whole neighborhood $\Upsilon \subset \Omega$ of the point (ξ_L^0, ξ_R^0) then the surface doesn't exist in this neighborhood Υ .

Using (3.7) we can write the formula for Gaussian curvature¹ as

$$K = \frac{1}{\sqrt{J_L J_R - G_{LR}^2}} \partial_R \left(\frac{\partial_L G_{LR} - (1/2)G_{LR} \partial_L (\ln J_L)}{\sqrt{J_L J_R - G_{LR}^2}} \right), \tag{3.14}$$

where

$$G_{LR} = -\Re \left(\frac{\partial_R f^\dagger P\partial_L f}{f^\dagger f} \right).$$

In the $\mathbb{C}P^1$ case a surprising simplification occurs and we find that the Gaussian curvature is a negative constant, $K = -4$. Consequently, there are no umbilical points on the surface and any regular solution of the Euler–Lagrange equations (2.12) gives rise to a pseudosphere immersed in $su(2) \simeq \mathbb{R}^3$. Several examples of such pseudospheres were present in [31], one is also reproduced in Section 6.

4. The Gauss–Weingarten equations

Now we may formally determine a moving frame on the surface \mathcal{F} and write the Gauss–Weingarten equations in the $\mathbb{C}P^{N-1}$ case. Let f be a solution of the Euler–Lagrange equa-

¹ By Gaussian curvature we shall understand the half of scalar curvature, i.e. of the fully contracted Riemann curvature tensor $K = (1/2)R = (1/2)R_{BD}^{BD}$ is derived from the induced metric G . Such definition of the Gaussian curvature coincides with its definition in terms of principal curvatures in \mathbb{R}^3 .

tions (2.12) such that $\det(G)$ is not zero in a neighborhood of a regular point (ξ_L^0, ξ_R^0) in Ω . Assume also that the surface \mathcal{F} (3.6), associated with these equations is described by the moving frame

$$\vec{\tau} = (\partial_L X, \partial_R X, n_3, \dots, n_{N^2-1})^T,$$

where the vectors $\partial_L X, \partial_R X, n_3, \dots, n_{N^2-1}$ satisfy the normalization conditions

$$\begin{aligned} (\partial_L X, \partial_L X) &= J_L, \\ (\partial_L X, \partial_R X) &= G_{LR}, \\ (\partial_R X, \partial_R X) &= J_R, \\ (\partial_L X, n_k) &= (\partial_R X, n_k) = 0, \\ (n_j, n_k) &= \delta_{jk}. \end{aligned} \tag{4.1}$$

We now show that the moving frame satisfies the Gauss–Weingarten equations

$$\begin{aligned} \partial_L \partial_L X &= A_L^L \partial_L X + A_R^L \partial_R X + Q_j^L n_j, \\ \partial_L \partial_R X &= \tilde{H}_j n_j, \\ \partial_L n_j &= \alpha_j^L \partial_L X + \beta_j^L \partial_R X + s_{jk}^L n_k, \\ \partial_R \partial_L X &= \tilde{H}_j n_j, \\ \partial_R \partial_R X &= A_L^R \partial_L X + A_R^R \partial_R X + Q_j^R n_j, \\ \partial_R n_j &= \alpha_j^R \partial_L X + \beta_j^R \partial_R X + s_{jk}^R n_k, \end{aligned} \tag{4.2}$$

where $s_{jk}^L + s_{kj}^L = 0, s_{jk}^R + s_{kj}^R = 0, j, k = 3, \dots, N^2 - 1$,

$$\begin{aligned} \alpha_j^L &= \frac{\tilde{H}_j G_{LR} - Q_j^L J_R}{\det G}, & \beta_j^L &= \frac{Q_j^L G_{LR} - \tilde{H}_j J_L}{\det G}, \\ \alpha_j^R &= \frac{Q_j^R G_{LR} - \tilde{H}_j J_R}{\det G}, & \beta_j^R &= \frac{\tilde{H}_j G_{LR} - Q_j^R J_L}{\det G}, \end{aligned}$$

$$\begin{aligned} A_L^L &= \frac{1}{\det G} \Re \left\{ \frac{1}{f^\dagger f} (J_R \partial_L f^\dagger + G_{LR} \partial_R f^\dagger) P \partial_L \partial_L f - \frac{2 \partial_L f^\dagger f}{(f^\dagger f)^2} (\partial_L f^\dagger P \partial_R f) G_{LR} \right. \\ &\quad \left. - \frac{2 f^\dagger \partial_L f}{f^\dagger f} J_L J_R \right\}, \\ A_R^L &= \frac{1}{\det G} \Re \left\{ -\frac{1}{f^\dagger f} (J_L \partial_R f^\dagger + G_{LR} \partial_L f^\dagger) P \partial_L \partial_L f + \frac{2 \partial_L f^\dagger f}{(f^\dagger f)^2} (\partial_L f^\dagger P \partial_R f) J_L \right. \\ &\quad \left. + \frac{2 f^\dagger \partial_L f}{f^\dagger f} J_L G_{LR} \right\}, \end{aligned} \tag{4.3}$$

and A_L^R, A_R^R have similar form which can be obtained by exchange $L \leftrightarrow R$. The explicit form of the coefficients \tilde{H}_j, Q_j^D (where $D = L, R; j = 3, \dots, N^2 - 1$) depends on the chosen orthonormal basis $\{n_3, \dots, n_{N^2-1}\}$ of the normal space to the surface \mathcal{F} at the point $X(\xi_L^0, \xi_R^0)$.

Indeed, if $\partial_L X$ and $\partial_R X$ are defined by (3.4) for an arbitrary solution f of the Euler–Lagrange equations (2.12), then by straightforward calculation using (2.12) one finds that

$$\begin{aligned} \partial_L \partial_R X = \partial_R \partial_L X = [\partial_L P, \partial_R P] &= \frac{1}{f^\dagger f} (P \partial_L f \otimes \partial_R f^\dagger P - P \partial_R f \otimes \partial_L f^\dagger P) \\ &+ \frac{1}{(f^\dagger f)^2} (\partial_L f^\dagger P \partial_R f - \partial_R f^\dagger P \partial_L f) f \otimes f^\dagger. \end{aligned} \tag{4.4}$$

By computing

$$\text{tr}(\partial_L \partial_R X \cdot \partial_D X) = \pm \text{tr}([\partial_L P, \partial_R P] \cdot [\partial_D P, P]) = 0, \quad D = L, R \tag{4.5}$$

we conclude that $\partial_L \partial_R X$ is perpendicular to the surface \mathcal{F} and consequently it has the form given in (4.2).

The remaining relations in (4.2) and (4.3) follow as differential consequences from the assumed normalizations of the normals (4.1), e.g.

$$(n_j, n_k) = 0, \quad j \neq k$$

which gives,

$$0 = (\partial_L n_j, n_k) + (\partial_L n_k, n_j) = s_{jk}^L + s_{kj}^L.$$

Similarly

$$(n_j, \partial_L X) = 0, \quad (n_j, \partial_R X) = 0$$

by differentiation leads to

$$(\partial_R n_j, \partial_L X) + (n_j, \partial_L \partial_R X) = 0, \quad (\partial_R n_j, \partial_R X) + (n_j, \partial_R \partial_R X) = 0$$

implying

$$J_L \alpha_j^R + G_{LR} \beta_j^R + \tilde{H}_j = 0, \quad G_{LR} \alpha_j^R + J_R \beta_j^R + Q_j^R = 0.$$

Consequently, α_j^R and β_j^R can be determined in terms of \tilde{H}_j, Q_j^R and of the components of the induced metric G . The remaining coefficients α_j^L and β_j^L are derived in an analogous way by exchanging indices $L \leftrightarrow R$ in the successive differentiations.

The coefficients A_L^L, \dots, A_R^R are obtained by requiring that $(\partial_D \partial_D X - A_L^D \partial_L X - A_R^D \partial_R X)$ is normal to the surface, i.e.

$$\text{tr}(\partial_B X \cdot (\partial_D \partial_D X - A_L^D \partial_L X - A_R^D \partial_R X)) = 0, \quad B, D = L, R. \tag{4.6}$$

From (3.2) and (3.5) we find

$$\begin{aligned} \partial_L \partial_L X &= \frac{1}{f^\dagger f} (P \partial_L \partial_L f \otimes f^\dagger - f \otimes \partial_L \partial_L f^\dagger P) + \frac{2}{(f^\dagger f)^2} ((\partial_L f^\dagger f) f \otimes \partial_L f^\dagger P \\ &\quad - (f^\dagger \partial_L f) P \partial_L f \otimes f^\dagger), \\ \partial_R \partial_R X &= \frac{1}{f^\dagger f} (f \otimes \partial_R \partial_R f^\dagger P - P \partial_R \partial_R f \otimes f^\dagger) + \frac{2}{(f^\dagger f)^2} ((f^\dagger \partial_R f) P \partial_R f \otimes f^\dagger \\ &\quad - (\partial_R f^\dagger f) f \otimes \partial_R f^\dagger P), \end{aligned} \tag{4.7}$$

and after substituting the above expressions into (4.6) we solve the resulting linear equations for A_B^D .

Let us note that the Gauss–Weingarten equations (4.2) can be written equivalently in the $N \times N$ matrix form

$$\partial_L \vec{\tau} = U \vec{\tau}, \quad \partial_R \vec{\tau} = V \vec{\tau}, \tag{4.8}$$

where

$$\begin{aligned} U &= \begin{pmatrix} A_L^L & A_R^L & Q_3^L & \cdots & \cdots & Q_{N^2-1}^L \\ 0 & 0 & \tilde{H}_3 & \cdots & \cdots & \tilde{H}_{N^2-1} \\ \alpha_3^L & \beta_3^L & 0 & s_{34}^L & \cdots & s_{3(N^2-1)}^L \\ \alpha_4^L & \beta_4^L & -s_{34}^L & 0 & \cdots & s_{4(N^2-1)}^L \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{(N^2-1)}^L & \beta_{(N^2-1)}^L & -s_{3(N^2-1)}^L & \cdots & -s_{(N^2-2)(N^2-1)}^L & 0 \end{pmatrix}, \\ V &= \begin{pmatrix} 0 & 0 & \tilde{H}_3 & \cdots & \cdots & \tilde{H}_{N^2-1} \\ A_L^R & A_R^R & Q_3^R & \cdots & \cdots & Q_{N^2-1}^R \\ \alpha_3^R & \beta_3^R & 0 & s_{34}^R & \cdots & s_{3(N^2-1)}^R \\ \alpha_4^R & \beta_4^R & -s_{34}^R & 0 & \cdots & s_{4(N^2-1)}^R \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{(N^2-1)}^R & \beta_{(N^2-1)}^R & -s_{3(N^2-1)}^R & \cdots & -s_{(N^2-2)(N^2-1)}^R & 0 \end{pmatrix}. \end{aligned} \tag{4.9}$$

The Gauss–Codazzi–Ricci equations

$$\partial_R U - \partial_L V + [U, V] = 0 \tag{4.10}$$

are compatibility conditions for the Gauss–Weingarten equations (4.2). They are the necessary and sufficient conditions for the local existence of the corresponding surface \mathcal{F} . It can be easily checked that they are identically satisfied for any solution f of the Euler–Lagrange equations (2.12).

The second fundamental form of the surface \mathcal{F} at the regular point p takes in general² the shape of a map

$$\mathbf{II}(p) : T_p\mathcal{F} \times T_p\mathcal{F} \rightarrow N_p\mathcal{F},$$

where $T_p\mathcal{F}$ and $N_p\mathcal{F}$ denote the tangent and normal space to the surface \mathcal{F} at the point p , respectively. According to [32,33], the second fundamental form and the mean curvature vector can be expressed as

$$\mathbf{II} = (\partial_L\partial_L X)^\perp d\xi_L d\xi_L + 2(\partial_L\partial_R X)^\perp d\xi_L d\xi_R + (\partial_R\partial_R X)^\perp d\xi_R d\xi_R, \tag{4.11}$$

$$\mathbf{H} = \frac{1}{\det G} (J_R(\partial_L\partial_L X)^\perp - 2G_{LR}(\partial_L\partial_R X)^\perp + J_L(\partial_R\partial_R X)^\perp), \tag{4.12}$$

where $(\cdot)^\perp$ denotes the normal part of the vector. In our case, given the decomposition of $\partial_D\partial_B X$ into the tangent and normal parts in the Gauss–Weingarten equations (4.2), the expressions (4.11) and (4.12) take the form

$$\begin{aligned} \mathbf{II} &= (\partial_L\partial_L X - A_L^L\partial_L X - A_R^L\partial_R X) d\xi_L d\xi_L + 2(\partial_L\partial_R X) d\xi_L d\xi_R \\ &\quad + (\partial_R\partial_R X - A_L^R\partial_L X - A_R^R\partial_R X) d\xi_R d\xi_R, \end{aligned} \tag{4.13}$$

$$\begin{aligned} \mathbf{H} &= \frac{1}{\det G} (J_R(\partial_L\partial_L X - A_L^L\partial_L X - A_R^L\partial_R X) - 2G_{LR}(\partial_L\partial_R X) \\ &\quad + J_L(\partial_R\partial_R X - A_L^R\partial_L X - A_R^R\partial_R X)). \end{aligned} \tag{4.14}$$

Consequently, the Willmore functional [33,5] is

$$W_{\mathcal{F}} = \int |\mathbf{H}|^2 \sqrt{\det G} d\xi_L d\xi_R. \tag{4.15}$$

The derivatives $\partial_D\partial_B X$ are expressed explicitly in terms of f in Eqs. (4.4) and (4.7). Unfortunately, it is clear that after explicit calculation of $(\partial_B\partial_D X)^\perp$ in the case of $N > 2$, both the second fundamental form and the mean curvature vector contain terms like $P\partial_L\partial_L f \otimes f^\dagger$, etc., which are neither cancelled out by other terms nor projected out by the normal projection. Therefore, the resulting expressions are rather complicated and, for lack of space, we do not present them here.

In the $\mathbb{C}P^1$ case the formulae (4.13) and (4.14) simplify to

$$\mathbf{II} = -2(\partial_R f^\dagger P\partial_L f - \partial_L f^\dagger P\partial_R f)(\mathbf{1} - 2P) d\xi_L d\xi_R,$$

$$\mathbf{H} = 2 \left(\frac{\partial_R f^\dagger P\partial_L f + \partial_L f^\dagger P\partial_R f}{\partial_R f^\dagger P\partial_L f - \partial_L f^\dagger P\partial_R f} \right) (\mathbf{1} - 2P),$$

where the normal to the surface \mathcal{F} is given by

$$n = i(\mathbf{1} - 2P) \in su(2). \tag{4.16}$$

² In the familiar \mathbb{R}^3 case the normal space $N_p\mathcal{F}$ is conventionally identified with \mathbb{R} .

5. The moving frame of a surface in the algebra $su(N)$

Now we proceed to construct the moving frame of the surface \mathcal{F} immersed in $su(N)$ algebra, i.e. matrices $\partial_L X, \partial_R X, n_a, a = 3, \dots, N^2 - 1$ satisfying (4.1).

Let f be a solution of the Euler–Lagrange equations (2.12) and let (ξ_L^0, ξ_R^0) be a regular point in Ω , i.e. such that $\det G(f(\xi_L^0, \xi_R^0)) \neq 0$. Let us denote $f^0 = f(\xi_L^0, \xi_R^0)$, $X^0 = X(\xi_L^0, \xi_R^0)$. Taking into account that

$$\text{tr}(A) = \text{tr}(\Phi A \Phi^\dagger), \quad A \in su(N), \Phi \in SU(N),$$

we employ the adjoint representation of the group $SU(N)$ in order to bring $\partial_L X, \partial_R X, n_a$ to the simplest form possible. We shall request

$$\Phi^\dagger f^0 = (\sqrt{f^{0\dagger} f^0}, 0, \dots, 0)^T. \tag{5.1}$$

Let us choose an orthonormal basis in $su(N)$ in the following form

$$\begin{aligned} (A_{jk})_{ab} &= i(\delta_{ja}\delta_{kb} + \delta_{jb}\delta_{ka}), \quad 1 \leq j < k \leq N, \\ (B_{jk})_{ab} &= (\delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka}), \quad 1 \leq j < k \leq N, \\ (C_p)_{ab} &= i\sqrt{\frac{2}{p(p+1)}} \left(\sum_{d=1}^p \delta_{da}\delta_{db} - p\delta_{p+1,a}\delta_{p+1,b} \right), \quad 1 \leq p \leq N - 1. \end{aligned} \tag{5.2}$$

The existence of $\Phi \in SU(N)$ satisfying (5.1) follows from the fact that the $SU(N)$ group acts transitively on the set $\{a \in \mathbb{C}^N, a^\dagger a = \alpha\}$, where $\alpha \in \mathbb{R}^+$. It should be noted that such Φ is not unique. A concrete form of Φ can be constructed as follows: starting from a general element $a = (a_1, \dots, a_N)^T$ of \mathbb{C}^N one firstly finds a transformation Φ_{N-1}^\dagger which transforms a into the vector

$$a^{(N-1)} = (a_1, \dots, a_{N-2}, \sqrt{a_{N-1}\bar{a}_{N-1} + a_N\bar{a}_N}, 0), \quad a^{(N-1)}\bar{a}^{(N-1)} = |a|^2.$$

It is easy to see that the desired transformation is

$$\Phi_{N-1}^\dagger = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \frac{\bar{a}_{N-1}}{(a_{N-1}\bar{a}_{N-1} + a_N\bar{a}_N)^{1/2}} & \frac{\bar{a}_N}{(a_{N-1}\bar{a}_{N-1} + a_N\bar{a}_N)^{1/2}} \\ 0 & \dots & -\frac{a_N}{(a_{N-1}\bar{a}_{N-1} + a_N\bar{a}_N)^{1/2}} & \frac{a_{N-1}}{(a_{N-1}\bar{a}_{N-1} + a_N\bar{a}_N)^{1/2}} \end{pmatrix}.$$

In the next step one eliminates the last but one element of $a^{(N-1)}$ using the transformation

$$\Phi_{N-2}^\dagger = \begin{pmatrix} 1 & 0 & & 0 & \dots & 0 \\ 0 & 1 & & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & \frac{\bar{a}_{N-2}}{\left(\sum_{j=N-2}^N a_j \bar{a}_j\right)^{1/2}} & \frac{\left(\sum_{j=N-1}^N a_j \bar{a}_j\right)^{1/2}}{\left(\sum_{j=N-2}^N a_j \bar{a}_j\right)^{1/2}} & 0 & 0 \\ 0 & \dots & -\frac{\left(\sum_{j=N-2}^N a_j \bar{a}_j\right)^{1/2}}{\left(\sum_{j=k}^N a_j \bar{a}_j\right)^{1/2}} & \frac{a_{N-2}}{\left(\sum_{j=N-2}^N a_j \bar{a}_j\right)^{1/2}} & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

This gives

$$a^{(N-2)} = \Phi_{N-2}^\dagger a^{(N-1)} = \Phi_{N-2}^\dagger \Phi_{N-1}^\dagger a = \left(a_1, \dots, a_{N-3}, \left(\sum_{j=N-2}^N a_j \bar{a}_j \right)^{1/2}, 0, 0 \right).$$

By induction and redesignation $a = f^0$ one arrives at the explicit form of Φ

$$\Phi_{N-1}^\dagger = \begin{pmatrix} 1 & 0 & & \dots & & 0 \\ 0 & 1 & & \dots & & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & \dots & \frac{\bar{f}_{N-1}^0}{\left(f_{N-1}^0 \bar{f}_{N-1}^0 + f_N^0 \bar{f}_N^0\right)^{1/2}} & & \frac{\bar{f}_N^0}{\left(f_{N-1}^0 \bar{f}_{N-1}^0 + f_N^0 \bar{f}_N^0\right)^{1/2}} & \\ 0 & \dots & -\frac{f_N^0}{\left(f_{N-1}^0 \bar{f}_{N-1}^0 + f_N^0 \bar{f}_N^0\right)^{1/2}} & & \frac{f_{N-1}^0}{\left(f_{N-1}^0 \bar{f}_{N-1}^0 + f_N^0 \bar{f}_N^0\right)^{1/2}} & \end{pmatrix},$$

$$\Phi = \Phi_{N-1} \Phi_{N-2} \dots \Phi_1, \quad \Phi^\dagger = \Phi_1^\dagger \Phi_2^\dagger \dots \Phi_{N-1}^\dagger \in SU(N), \tag{5.3}$$

where

$$\Phi_k^\dagger = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & \dots & \frac{\tilde{f}_k^0}{\left(\sum_{j=k}^N f_j^0 \tilde{f}_j^0\right)^{1/2}} & \frac{\left(\sum_{j=k+1}^N f_j^0 \tilde{f}_j^0\right)^{1/2}}{\left(\sum_{j=k}^N f_j^0 \tilde{f}_j^0\right)^{1/2}} & 0 & \dots \\ 0 & \dots & -\frac{\left(\sum_{j=k+1}^N f_j^0 \tilde{f}_j^0\right)^{1/2}}{\left(\sum_{j=k}^N f_j^0 \tilde{f}_j^0\right)^{1/2}} & \frac{f_k^0}{\left(\sum_{j=k}^N f_j^0 \tilde{f}_j^0\right)^{1/2}} & 0 & \dots \\ 0 & \dots & 0 & 0 & 1 & \dots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & 1 \end{pmatrix},$$

$k \leq N - 2.$ (5.4)

If any of the denominators vanishes then the corresponding matrix Φ_k is defined to be the unit matrix. It is also clear that the group element Φ constructed in this way is a smooth function of f, f^\dagger and consequently of ξ_L, ξ_R . Thus, we find

$$\begin{aligned} \Phi^\dagger f^0 &= (\sqrt{f^{0\dagger} f^0}, 0, \dots, 0)^T, \\ \partial_L^\Phi X^0 &\equiv \Phi^\dagger \partial_L X(\xi_L^0, \xi_R^0) \Phi = \frac{1}{\sqrt{f^{0\dagger} f^0}} \begin{pmatrix} 0 & -\partial_L^\Phi f^{0\dagger} \\ \partial_L^\Phi f^0 & \mathbf{0} \end{pmatrix}, \\ \partial_R^\Phi X^0 &\equiv \Phi^\dagger \partial_R X(\xi_L^0, \xi_R^0) \Phi = -\frac{1}{\sqrt{f^{0\dagger} f^0}} \begin{pmatrix} 0 & -\partial_R^\Phi f^{0\dagger} \\ \partial_R^\Phi f^0 & \mathbf{0} \end{pmatrix}, \end{aligned} \tag{5.5}$$

where $\mathbf{0}$ denotes the null $(N - 1) \times (N - 1)$ matrix and the vectors $\partial_D^\Phi f^0 \in \mathbb{C}^{N-1}$ are defined by

$$(\partial_D^\Phi f^0)_{j-1} = (\Phi^\dagger \partial_D f(\xi_L^0, \xi_R^0))_j, \quad D = L, R, \quad j = 2, \dots, N.$$

The construction of the moving frame is now straightforward. Assume that one finds, using a variant of Gram–Schmidt orthogonalization procedure, the orthonormal vectors

$$\tilde{A}_{1j}, \tilde{B}_{1j}, \quad j = 3, \dots, N$$

satisfying

$$(\partial_D^\Phi X^0, \tilde{A}_{1j}) = 0, \quad (\partial_D^\Phi X^0, \tilde{B}_{1j}) = 0$$

and

$$\text{span}(\partial_D^\Phi X^0, \tilde{A}_{1j}, \tilde{B}_{1j})_{D=L,R, j=3,\dots,N} = \text{span}(A_{1j}, B_{1j})_{j=2,\dots,N}. \tag{5.6}$$

We identify the remaining tilded and untilded matrices

$$\tilde{A}_{jk} = A_{jk}, \tilde{B}_{jk} = B_{jk}, \tilde{C}_p = C_p, \quad 1 < j < k \leq N, 1 \leq p \leq N - 1.$$

From (5.5) directly follows that

$$(\partial_D^\Phi X^0, \tilde{A}_{jk}) = (\partial_D^\Phi X^0, \tilde{B}_{jk}) = (\partial_D^\Phi X^0, \tilde{C}_p) = 0, \quad 1 < j < k \leq N, p < N$$

and as a result of Gramm–Schmidt orthogonalization we get

$$(\partial_D^\Phi X^0, \tilde{A}_{1k}) = (\partial_D^\Phi X^0, \tilde{B}_{1k}) = 0$$

and for $1 < j < k \leq N, p = 1, \dots, N - 1, i = 3, \dots, N$

$$(\tilde{A}_{1i}, \tilde{A}_{jk}) = (\tilde{A}_{1i}, \tilde{B}_{jk}) = (\tilde{A}_{1i}, \tilde{C}_p) = (\tilde{B}_{1i}, \tilde{A}_{jk}) = (\tilde{B}_{1i}, \tilde{B}_{jk}) = (\tilde{B}_{1i}, \tilde{C}_p) = 0.$$

Therefore, under the above given assumptions and notation, we can state the following

Proposition 1. *The moving frame of the surface \mathcal{F} at the point $X^0 = X(\xi_L^0, \xi_R^0)$*

$$\begin{aligned} \partial_L X &= \Phi \partial_L^\Phi X^0 \Phi^\dagger, \\ \partial_R X &= \Phi \partial_R^\Phi X^0 \Phi^\dagger, \\ n_{jk}^A &= \Phi \tilde{A}_{jk} \Phi^\dagger, \\ n_{jk}^B &= \Phi \tilde{B}_{jk} \Phi^\dagger, \quad 1 \leq j < k \leq N, \\ n_p^C &= \Phi \tilde{C}_p \Phi^\dagger, \quad 1 \leq p \leq N - 1. \end{aligned} \tag{5.7}$$

satisfies the normalization conditions (4.1) and consequently the Gauss–Weingarten equations (4.2).

Note that the first two lines of (5.7) are equivalent to (5.5). The remaining lines of (5.7) give a rather explicit description of normals to the surface \mathcal{F} . In the $\mathbb{C}P^1$ case a significant simplification occurs, namely there is only one normal vector $n_1^C = i\Phi\sigma_3\Phi^{-1}$ to the surface immersed in $su(2)$ and no orthogonalization is needed.

In the case of $N > 2$ the explicit form of the moving frame (5.7) might be quite complicated because of the orthogonalization process involved in the construction of

$$n_{1j}^A, n_{1j}^B, \quad j = 3, \dots, N$$

(i.e. in the construction of $\tilde{A}_{1j}, \tilde{B}_{1j}, j = 3, \dots, N$). On the other hand, the remaining normals

$$n_{jk}^A, n_{jk}^B, n_p^C, \quad 1 < j < k \leq N, 1 \leq p \leq N - 1,$$

can be constructed without any difficulty. If we chose other group element Φ satisfying (5.1), the constructed normals would have been rotated by a local (gauge) transformation from the subgroup of $SU(N)$ leaving $\partial_L X(\xi_L^0, \xi_R^0), \partial_R X(\xi_L^0, \xi_R^0)$ invariant.

It is also worth noting that from the Eqs. (4.4) and (4.7) immediately follows that

$$\begin{aligned} \partial_L \partial_L X, \partial_R \partial_R X &\in \Phi \operatorname{span}(A_{1j}, B_{1j})_{j=2, \dots, N} \Phi^\dagger, \\ \partial_L \partial_R X &\in \Phi \operatorname{span}(A_{jk}, B_{jk}, C_p)_{1 < j < k \leq N, p < N} \Phi^\dagger, \end{aligned} \tag{5.8}$$

i.e.

$$\begin{aligned} (\partial_L \partial_L X)^\perp, (\partial_R \partial_R X)^\perp &\in \operatorname{span}(n_{1j}^A, n_{1j}^B)_{j=3, \dots, N}, \\ (\partial_L \partial_R X)^\perp &= \partial_L \partial_R X \in \operatorname{span}(n_{jk}^A, n_{jk}^B, n_p^C)_{1 < j < k \leq N, p < N}. \end{aligned} \tag{5.9}$$

Concerning other possible constructions of the normals, one can observe that one may construct immediately two unit normals³

$$n_P = i\sqrt{2} \left(\sqrt{\frac{N-1}{N}} \mathbf{1} - \sqrt{\frac{N}{N-1}} P \right),$$

$$n_{[\partial_L X, \partial_R X]} = \frac{i[\partial_L X, \partial_R X]}{||[\partial_L X, \partial_R X]||}.$$

In the $su(2)$ case the normals $n_P, n_{[\partial_L X, \partial_R X]}, n_1^C$ coincide up to the choice of orientation, but in general the relation of $n_P, n_{[\partial_L X, \partial_R X]}$ to $n_{jk}^A, n_{jk}^B, n_p^C$ is rather complicated and difficult to express in a closed form. In principle one could attempt to construct the moving frame directly from these normals by taking normal parts of commutators of them with $\partial_L X$ and $\partial_R X$, etc.,⁴ without need to construct the group element Φ . Unfortunately, such procedure does not seem to be computationally feasible at the moment, leaving this subject open for further investigation.

6. Example of surface in the algebra $su(2)$

As an example of a surface obtained using the described method we present a picture of a surface in $su(2)$ associated with the CP^1 sigma model on Minkowski space (see Fig. 1).

³ Note that the scalar product

$$(n_P, n_{[\partial_L X, \partial_R X]}) \neq 0$$

so that their orthogonalization would be needed.

⁴ This can be proved to be possible at least in the $su(3)$ case by observing that $\partial_L X$ and $\partial_R X$ generate via commutators the whole algebra $su(3)$.

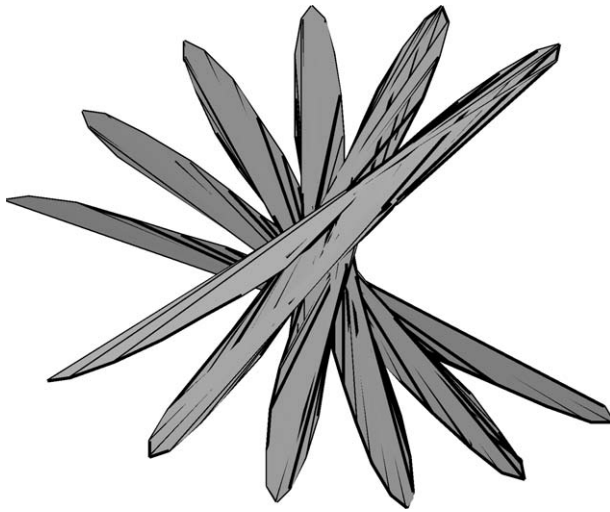


Fig. 1. The surface associated with the solution (6.1), $p = -3/2$.

The following solution of the Euler–Lagrange equations (2.12) in this case was obtained by us using the symmetry reduction method

$$f = \left(1, \sqrt{\frac{(p-1)\cosh(g(\chi)) + (p+1)}{(p-1)\cosh(g(\chi)) - (p+1)}} \exp(i(\xi_L - h(\chi))) \right), \tag{6.1}$$

where

$$h(\chi) = \arctan\left(\frac{p+1}{2\sqrt{-p}} \tanh g(\chi)\right) + \frac{(p+2\sqrt{-p}-1)\chi}{2(p-1)},$$

$$g(\chi) = \frac{(p+1)\chi}{2(p-1)}, \quad \chi = \xi_L - \xi_R, \quad p < -1.$$

The formulae for the first and second fundamental forms, moving frame, etc., of the associated surface \mathcal{F} were obtained but are too lengthy to be presented here. The computation of the surface, i.e. the Weierstrass representation (3.6), was performed numerically. The Gaussian curvature is $K = -4$, the mean curvature is

$$H = -\frac{e^{4g(\chi)} - 6e^{2g(\chi)} + 1}{2e^{g(\chi)}(e^{2g(\chi)} - 1)}.$$

Other examples were presented in [31].

7. Conclusions

The main purpose of this paper was to provide the structural equations of two-dimensional orientable smooth surfaces immersed in $su(N)$ algebra. The surfaces were obtained from the $\mathbb{C}P^{N-1}$ sigma model defined on two-dimensional Minkowski space.

The most important advantage of the method presented is that it gives effective tools for constructing surfaces without reference to additional considerations, proceeding directly from the given $\mathbb{C}P^{N-1}$ model equations (2.12). We demonstrated through the use of Cartan's language of moving frames that one can derive via $\mathbb{C}P^{N-1}$ model the first and second fundamental forms of the surface as well as the relations between them as expressed in the Gauss–Weingarten and Gauss–Codazzi–Ricci equations. We presented an extension of the classical Enneper–Weierstrass representation [29,2] of surfaces in multi-dimensional spaces, expressed in terms of any nonsingular (i.e. such that $\det G \neq 0$) solution of the $\mathbb{C}P^{N-1}$ sigma model.

In particular, we showed that in the $\mathbb{C}P^1$ case such description of surfaces in the $su(2)$ algebra leads to constant negative Gaussian curvature surfaces. Such surfaces were considered in relation to sin-Gordon equation in [34] and we shall investigate the relation between the $\mathbb{C}P^1$ sigma models and sin-Gordon equation and their corresponding surfaces in a separate paper. In higher dimensional cases the Gaussian curvature is no longer constant. Therefore, another direction of our future research is the construction of nontrivial solutions of $\mathbb{C}P^{N-1}$ sigma models and their corresponding surfaces. Investigation of concrete examples should help to reveal which geometric properties account for the fact that higher dimensional models admit more diverse surfaces, namely with nonconstant Gaussian curvature, than the $\mathbb{C}P^1$ case.

A different direction in the search for characterizing geometric properties is to study global quantities like the Willmore functional $W_{\mathcal{F}}$ (4.15) and ask whether surfaces associated to $\mathbb{C}P^{N-1}$ sigma model correspond to some special value of $W_{\mathcal{F}}$. We checked explicitly on one constructed surface in $su(2)$ whether it satisfies the local condition for the extremum of $W_{\mathcal{F}}$

$$\Delta_{\mathcal{F}}H + 2H(H^2 - K) = 0$$

and it doesn't. Evaluating the Willmore functional meets with difficulties because surfaces associated to models on Minkowski space are usually non-compact—as is evident in the $\mathbb{C}P^1$ case, due to the constant negative Gaussian curvature. Consequently, all global invariants, variational principles, etc., depend on boundary conditions imposed on the solution which further complicates their study. On the other hand, such methods might be more fruitful in the case of models on Euclidean space [15,16] where surfaces are compact provided the solutions can be extended to the compactification S^2 of Euclidean \mathbb{R}^2 .

Concerning possible applications of our method we assume that it can be particularly useful in the theory of phase transitions or fluid membranes [17,18,25–27], where numerical approaches have prevailed so far. Even in cases when the Weierstrass representation of a surface cannot be integrated explicitly, the surface's main characteristics can be derived in analytical form which lends itself to physical interpretations.

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